Lecture 4: Quantum states of light — Fock states

- Definition Fock states, photons

- Photon statistics of quantized light prepared in Fock states

<u>Fock states, definition</u>: So far, we have concentrated on introducing operators for the vector potential and thus the electric field. We have found that the quantized free electromagnetic field is an infinite collection of uncoupled harmonic oscillators, each of which is described by a Hamiltonian $\hat{H}_\lambda = \hbar\omega_\lambda(\hat{n}_\lambda + 1/2)$. Since all oscillators are uncoupled, it is sufficient to study a single oscillator. The extension to multi-mode systems is then straightforward. From now on we will omit the mode index λ.

We will now concentrate on the quantum states of light. The eigenvalue problem for $\hat{H} = \hbar\omega(\hat{n} + 1/2)$ is solved by considering the eigenvalue problem for the number operator since $[\hat{H}, \hat{n}] = 0$. The solution is known from quantum mechanics, and we will briefly review it here. The eigenvalue equation reads

$$\hat{n}|\psi_n\rangle = n|\psi_n\rangle \tag{4.1}$$

where n is the eigenvalue of $\hat{n}$ and $|\psi_n\rangle$ the corresponding eigenvector. The number operator $\hat{n} = \hat{a}^\dagger\hat{a}$ is an Hermitian operator, therefore its eigenvalues n are real and its eigenvectors $|\psi_n\rangle$ form a complete set of orthogonal states. Writing

$$\langle\psi_n|\hat{n}|\psi_n\rangle = \langle\psi_n|\hat{a}^\dagger\hat{a}|\psi_n\rangle = n\langle\psi_n|\psi_n\rangle \,, \tag{4.2}$$

we see that n must be a non-negative number. This is because $|\psi_n\rangle$ and $\hat{a}|\psi_n\rangle$ are both Hilbert-space vectors with non-negative norms. If we apply the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, we find that $[\hat{a}, \hat{n}] = [\hat{a}, \hat{a}^\dagger\hat{a}] = \hat{a}[\hat{a}, \hat{a}^\dagger] = \hat{a}$, and hence

$$\hat{n}\hat{a}|\psi_n\rangle = (\hat{a}\hat{n} - [\hat{a}, \hat{n}])|\psi_n\rangle = (n-1)\hat{a}|\psi_n\rangle \,. \tag{4.3}$$

This means that $\hat{a}|\psi_n\rangle$ is an eigenvector of the number operator with eigenvalue $n-1$ which we can call $|\psi_{n-1}\rangle$. But since all eigenvalues of $\hat{n}$ must be non-negative, there has to be a state $|\psi_0\rangle$ with $\hat{a}|\psi_0\rangle = 0$. This state will be called the ground state. It contains no excitation, and thus no quantum can be further annihilated. Similarly, we find that

$$\hat{n}\hat{a}^\dagger|\psi_n\rangle = (n+1)\hat{a}^\dagger|\psi_n\rangle \,. \tag{4.4}$$

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From Eqs. (4.3) and (4.4) we see that $\hat{a}$ and $\hat{a}^\dagger$ lower and raise the number of excitations by one, hence their names annihilation and creation operators.

Starting from the ground state $|\psi_0\rangle$, all other states can be generated by successive application of the creation operator, $|\psi_n\rangle = (\hat{a}^\dagger)^n|\psi_0\rangle$. If we normalize these states according to $|n\rangle = |\psi_n\rangle/\langle\psi_n|\psi_n\rangle$, we arrive at the definition of the *number states* or *Fock states*. We assume that the ground state (or vacuum state) $|0\rangle \equiv |\psi_0\rangle$ is already normalized. Then, we create the nth Fock state by

$$|n\rangle = c_n \left(\hat{a}^\dagger\right)^n |0\rangle \tag{4.5}$$

where we have yet to determine the normalization constants c_n. This is done by looking at the normalization condition

$$\langle n|n\rangle = |c_n|^2 \langle 0|\hat{a}^n \left(\hat{a}^\dagger\right)^n |0\rangle = 1 \,. \tag{4.6}$$

We then calculate

$$\langle 0|\hat{a}^n \left(\hat{a}^\dagger\right)^n |0\rangle = \langle 0|\hat{a}^{n-1}\hat{a}\hat{a}^\dagger \left(\hat{a}^\dagger\right)^{n-1} |0\rangle = \langle 0|\hat{a}^{n-1} \left(\hat{a}^\dagger\right)^{n-1} |0\rangle + \langle 0|\hat{a}^{n-1}\hat{n} \left(\hat{a}^\dagger\right)^{n-1} |0\rangle$$

and use the commutation rule $[\hat{n}, (\hat{a}^\dagger)^k] = k(\hat{a}^\dagger)^k$ to obtain

$$\langle 0|\hat{a}^n \left(\hat{a}^\dagger\right)^n |0\rangle = n\langle 0|\hat{a}^{n-1} \left(\hat{a}^\dagger\right)^{n-1} |0\rangle = n(n-1)\langle 0|\hat{a}^{n-2} \left(\hat{a}^\dagger\right)^{n-2} |0\rangle = \ldots = n!\langle 0|0\rangle$$

from which it follows that $c_n = 1/\sqrt{n!}$. Collecting all the results, we find that

$$\boxed{|n\rangle = \frac{1}{\sqrt{n!}} \left(\hat{a}^\dagger\right)^n |0\rangle} \,. \tag{4.7}$$

Similarly, we find that

$$\boxed{\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \,, \qquad \hat{a}|n\rangle = \sqrt{n}|n-1\rangle} \,. \tag{4.8}$$

As mentioned before, the eigenvectors of the number operator are mutually orthogonal, hence

$$\langle m|n\rangle = \delta_{mn} \,,$$

and they form a complete set of orthonormal vectors, hence

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{I} \,, \tag{4.9}$$

where $\hat{I}$ is the identity operator in the Hilbert space of the single-mode system.

<u>Photons</u>: The quantization of Maxwell's equations thus implies the existence of elementary excitations of the electromagnetic field with quantized energy $\hbar\omega_\lambda$. These excitations, or particles, are the *photons*. Hence, the operator $\hat{a}_\lambda^\dagger$ creates a photon of mode λ whereas the operator $\hat{a}_\lambda$ destroys a photon of energy $\hbar\omega_\lambda$ from the mode λ.

<u>Photon statistics of quantized light prepared in Fock states</u>: We will now concentrate on the properties of the Fock states. Because a single-mode system in the Fock state $|n\rangle$ contains exactly n excitations of energy $\hbar\omega$, they describe photons in a number state with exactly n quanta. The average energy of such a single-mode system is

$$\langle n|\hat{H}|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right) \tag{4.10}$$

so that indeed an energy $\hbar\omega$ is associated with each photon. Note that even if there is no photon present ($n = 0$), the vacuum energy of the harmonic oscillator is $\langle 0|\hat{H}|0\rangle = \hbar\omega/2 > 0$. Since there is a vacuum energy of $\hbar\omega/2$ associated with each mode, the ground-state energy of the free electromagnetic field is infinite. However, although being infinite, the vacuum energy is constant, and all energies are counted with respect to this infinite background. Only when the mode structure of the electromagnetic field and hence the vacuum energy is altered due to the presence of macroscopic bodies, the resulting finite change of the infinite energy can yield a force acting on the macroscopic bodies, the *Casimir force* (see previous lecture).

The fluctuations of an observable $\hat{O}$ are measured by the variance $\langle(\Delta\hat{O})^2\rangle$, where $\Delta\hat{O} = \hat{O} - \langle\hat{O}\rangle$. For the photon-number fluctuations of a single-mode radiation field in a Fock state we calculate

$$\langle n|(\Delta\hat{n})^2|n\rangle = \langle n|\hat{n}^2|n\rangle - \langle n|\hat{n}|n\rangle^2 = 0\,. \tag{4.11}$$

The fluctuations must vanish because the value of the photon number is 'sharp' in a Fock state. For the same reason, there are no energy fluctuations in a Fock state.

A more striking feature of the Fock states is revealed when we look at the expectation value of the electric field. From Eq. (2.13) and the relation $\hat{\mathbf{E}}(\mathbf{r}) = -\dot{\hat{\mathbf{A}}}(\mathbf{r})$ we see that for a single-mode field

$$\hat{\mathbf{E}}(\mathbf{r}) = \hat{\mathbf{E}}^{(+)}(\mathbf{r}) + \hat{\mathbf{E}}^{(-)}(\mathbf{r}) = i\omega\left[\mathbf{A}(\mathbf{r})\hat{a} - \mathbf{A}^*(\mathbf{r})\hat{a}^\dagger\right]\,, \tag{4.12}$$

where $\mathbf{A}(\mathbf{r})$ is again the classical mode function and $\hat{\mathbf{E}}^{(\pm)}(\mathbf{r})$ denotes the positive and negative

frequency parts of the electric field. Using the relations (4.8), we immediately see that

$$\langle n|\hat{\mathbf{E}}(\mathbf{r})|n\rangle = 0 \,. \tag{4.13}$$

That is, the expectation value of the electric field vanishes in a Fock state, although n photons are present. This is a rather unusual behaviour and shows that Fock states behave differently from their classical counterparts. In contrast, the expectation value of the intensity

$$\hat{I}(\mathbf{r}) = \hat{\mathbf{E}}^{(-)}(\mathbf{r}) \cdot \hat{\mathbf{E}}^{(+)}(\mathbf{r}) \tag{4.14}$$

in a photon-number state is proportional to the number of photons itself,

$$\langle n|\hat{I}(\mathbf{r})|n\rangle = \omega^2 |\mathbf{A}(\mathbf{r})|^2 n \,. \tag{4.15}$$

This is in line with our classical expectations. However, the fluctuations of the kth component of the electric field is

$$\langle n| \left[\Delta \hat{E}_k(\mathbf{r}) \right]^2 |n\rangle = \omega^2 |A_k(\mathbf{r})|^2 (2n + 1) \,. \tag{4.16}$$

Therefore, the fluctuations in the electric field increase with increasing photon number. Moreover, even if no photon is present ($n = 0$), there are zero-point fluctuations in the electric field,

$$\langle 0| \left[\Delta \hat{E}_k(\mathbf{r}) \right]^2 |0\rangle = \omega^2 |A_k(\mathbf{r})|^2 \,. \tag{4.17}$$

That is, not only does the vacuum state of the free electromagnetic field contain infinite energy, also the electric field fluctuates by an amount given by Eq. (4.17). Because of their highly nonclassical nature, Fock states are extremely hard to produce. For a survey
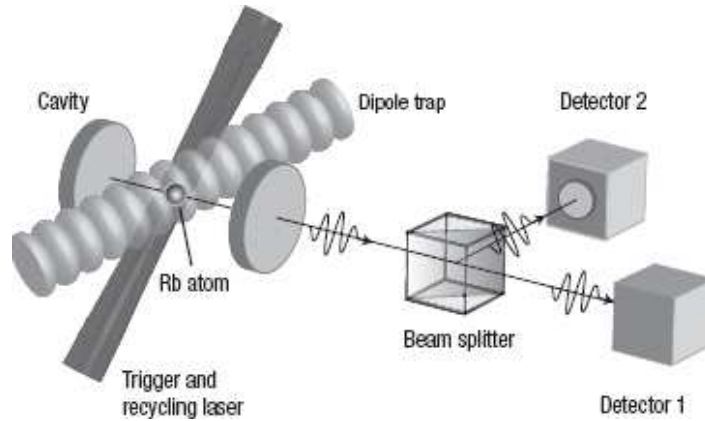
of experimental realizations and applications in quantum information processing (in a later lecture) and metrology, see M. Oxborrow and A.G. Sinclair, Contemp. Phys. **46**, 173 (2005).

Lecture 5: Quantum states of light — coherent and squeezed states

- coherent states and their statistical properties

- minimum-uncertainty states, squeezed states

Quantum states with classical analogues: Preparing photons in an energy eigenstate, or Fock state, leads to the peculiar effect that the mean value of the electric field vanishes, no matter how many photons are contained in the state. We would like to find other quantum states that resemble classical light more closely. In particular, we wish to find a quantum state in which the electric-field strength expectation value effectively takes its classical value

$$\langle\alpha|\hat{\mathbf{E}}(\mathbf{r})|\alpha\rangle = i\omega\left[\mathbf{A}(\mathbf{r})\alpha - \mathbf{A}^*(\mathbf{r})\alpha^*\right],$$

in which the photon annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ are replaced by the classical amplitudes α and α^*. The state $|\alpha\rangle$ will be called *coherent state*.

From quantum mechanics we know that one can transform a given complete set of orthonormal functions into another complete set by a unitary transformation. The unitary operator we will concentrate on for the moment is the diplacement operator

$$\hat{U} = \hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} \tag{5.1}$$

where α is a complex number. The displacement operator can be written in several operator orderings. The Baker–Campbell–Haussdorff formula for two operators $\hat{A}$ and $\hat{B}$ with $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$ can be written as (see supplementary notes)

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]} . \tag{5.2}$$

Applied to the displacement operator (5.1) with $\hat{A} = \alpha\hat{a}^\dagger$ and $\hat{B} = -\alpha^*\hat{a}$ this means that we obtain the normally and anti-normally ordered forms of the displacement operator as

$$\hat{D}(\alpha) = e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}e^{-|\alpha|^2/2} \qquad \text{normal order}, \tag{5.3}$$

$$\hat{D}(\alpha) = e^{-\alpha^*\hat{a}}e^{\alpha\hat{a}^\dagger}e^{|\alpha|^2/2} \qquad \text{anti-normal order}. \tag{5.4}$$

We now use the displacement operator to transform both the photonic amplitude operators as well as the Fock states as

$$\hat{a}' = \hat{D}(\alpha)\hat{a}\hat{D}^\dagger(\alpha), \tag{5.5}$$

$$|n\rangle' = \hat{D}(\alpha)|n\rangle . \tag{5.6}$$

In order to calculate the transformation of the operators, we use the expressions (5.3) and (5.4) to write $\hat{D}(\alpha)\hat{a}\hat{D}^\dagger(\alpha) = e^{\alpha\hat{a}^\dagger}\hat{a}e^{-\alpha\hat{a}^\dagger}$. Now we use the Baker–Haussdorff lemma which for arbitrary operators $\hat{A}$ and $\hat{B}$ states that

$$e^{z\hat{A}}\hat{B}e^{-z\hat{A}} = \sum_{n=0}^{\infty} \frac{z^n}{n!}\left[\hat{A}, \hat{B}\right]_n \tag{5.7}$$

($[\hat{A}, \hat{B}]_n = [\hat{A}, [\hat{A}, \hat{B}]_{n-1}]$, $[\hat{A}, \hat{B}]_0 = 1$). If we apply Eq. (5.7) with $\hat{B} \equiv \hat{a}$ and $z\hat{A} \equiv \alpha\hat{a}^\dagger$, we obtain $\hat{a}' = \hat{a} - \alpha$. If we apply this to the transformed ground state $|0\rangle'$, we get

$$\hat{a}'|0\rangle' \equiv 0 = (\hat{a} - \alpha)\hat{D}(\alpha)|0\rangle. \tag{5.8}$$

Now we call the transformed ground state $|0\rangle' = \hat{D}(\alpha)|0\rangle = |\alpha\rangle$ which depends on the complex number α, and we see from Eq. (5.8) that the coherent states $|\alpha\rangle$ are right-hand eigenstates of the non-Hermitian operator $\hat{a}$,

$$\boxed{\hat{a}|\alpha\rangle = \alpha|\alpha\rangle}. \tag{5.9}$$

The coherent states are normalized, $\langle\alpha|\alpha\rangle = 1$, and the amplitude α determines a point in the complex phase space corresponding to the coherent amplitude of the harmonic oscillator. However, as they are eigenstates of a non-Hermitian operator, they cannot be expected to be mutually orthogonal, nor will they satisfy a (simple) completeness relation. But we can use the completeness (4.9) of the Fock states to expand the coherent states into Fock states,

$$|\alpha\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}e^{-|\alpha|^2/2}|n\rangle. \tag{5.10}$$

Thus, the number distribution of photons in a coherent state is Poissonian, $|\langle\alpha|n\rangle|^2 = |\alpha|^{2n}e^{-|\alpha|^2}/n!$ with the mean value $\langle\alpha|\hat{n}|\alpha\rangle = |\alpha|^2$.

> Add-on: It turns out that the coherent states are over-complete. Using the expansion of the coherent states in terms of Fock states, the integral over the phase space yields
>
> $$\frac{1}{\pi}\int d^2\alpha\,|\alpha\rangle\langle\alpha| = \hat{I}. \tag{5.11}$$
>
> The relation (5.11) means that we can still expand every quantum state of light in terms of coherent states as they resolve the identity.

Let us now turn to the statistics of the coherent states. Indeed, the mean value of the kth component of the electric-field strength in a coherent state is

$$\langle\alpha|\hat{E}_k(\mathbf{r})|\alpha\rangle = i\omega\left[A_k(\mathbf{r})\alpha - A_k^*(\mathbf{r})\alpha^*\right].\tag{5.12}$$

Compared with the result for a photon-number state [Eq. (4.13)], the electric-field strength in a coherent state looks more like the electric field of a classical mode with coherent amplitude α, just as we have set out to show. The variance of the electric-field strength is

$$\langle\alpha|[\Delta\hat{E}_k(\mathbf{r})]^2|\alpha\rangle = \omega^2|A_k(\mathbf{r})|^2,\tag{5.13}$$

which is just the vacuum value [Eq. (4.17)] irrespective of the field amplitude. This also means that the relative noise of the electric field,

$$\sqrt{\frac{\langle\alpha|[\Delta\hat{E}_k(\mathbf{r})]^2|\alpha\rangle}{[\langle\alpha|\hat{E}_k(\mathbf{r})|\alpha\rangle]^2}} = \frac{1}{2|\alpha||\sin\varphi(A_k,\alpha)|},\tag{5.14}$$

decreases with the square root of the mean photon number, $|\alpha| = \sqrt{\langle\alpha|\hat{n}|\alpha\rangle}$. This behaviour is reminiscent of classical waves. Therefore, coherent states can be thought of as being the quantum states of light that resemble most closely classical coherent light.

Minimum-uncertainty states: In order to get a better picture of the coherent states in phase space, let us introduce the quadrature operator

$$\hat{x}(\varphi) = \hat{a}e^{i\varphi} + \hat{a}^\dagger e^{-i\varphi}\tag{5.15}$$

which depends on the angle φ. It is clear that we can write the operator of the electric field in terms of this quadrature operator as $\hat{E}_k(\mathbf{r}) = \omega|A_k(\mathbf{r})|\hat{x}(\varphi)$ with $\varphi = \arg A_k(\mathbf{r}) + \pi/2$. The quadrature operators for two different angles φ and φ' obey the commutation rule $[\hat{x}(\varphi),\hat{x}(\varphi')] = 2i\sin(\varphi - \varphi')$ as one can check using the commutation rule $[\hat{a},\hat{a}^\dagger] = 1$. Hence, two quadrature operators of orthogonal phase, $\varphi' = \varphi + \pi/2$, can be regarded as position and momentum operators in phase space [we have seen that already in Eq. (2.10)] with $[\hat{x}(\varphi),\hat{x}(\varphi+\pi/2)] = 2i$. Heisenberg's uncertainty principle implies that the product of the variances these quadratures satisfy the inequality [recall that $\Delta\hat{A}\Delta\hat{B} \geq |\langle[\hat{A},\hat{B}]\rangle|/2$]

$$\Delta\hat{x}(\varphi)\Delta\hat{x}(\varphi+\pi/2) \geq 1.\tag{5.16}$$

Going back to Eq. (5.13) and using the definition of the electric field in terms of the quadrature operator, we find that $\langle\alpha|[\Delta\hat{x}(\varphi)]^2|\alpha\rangle = 1$ for all values of φ. Hence, the coherent

states are *minimum-uncertainty states* because the uncertainty relation (5.16) obtains its lower limit. In other words, the coherent states cover a circular area in phase space of minimal allowed size (see Fig. 4).

Squeezed states: However, Heisenberg's uncertainty relation only limits the possible size that a quantum state has to cover in phase space. It says nothing about how the area should be shaped. We can think of compressing the uncertainty in one particular direction in phase space on the expense of increasing the noise in the orthogonal direction (see Fig. 4). Such quantum states do indeed exist and are called *squeezed states* or squeezed coherent states. They are minimum-uncertainty states like the coherent states, but show quantum noise below the vacuum noise in a particular direction in phase space. Such quantum states can be used in high-precision measurements as they beat the classical $1/\sqrt{n}$-limit.
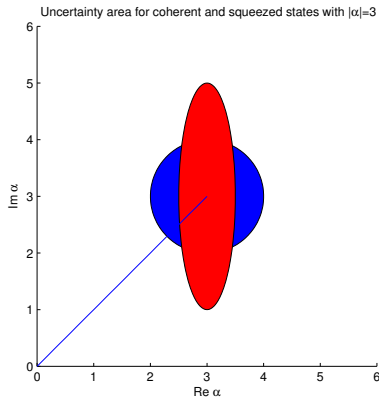

FIG. 4: Uncertainty area for a coherent state with $|\alpha| = 3$ and a squeezed state.

Squeezed states can be derived from coherent states by applying a unitary operator

$$\hat{U} = \hat{S}(\xi) = \exp\left\{\frac{1}{2}\left[\xi^*\hat{a}^2 - \xi\left(\hat{a}^\dagger\right)^2\right]\right\} \tag{5.17}$$

called the squeeze operator which depends on the squeezing parameter ξ. Such a unitary evolution operator can be realized with nonlinear materials as we will see later. The squeeze operator $\hat{S}(\xi)$ transforms the photonic amplitude operators as

$$\hat{a}' = \hat{S}(\xi)\hat{a}\hat{S}^\dagger(\xi) = \mu\hat{a} + \nu\hat{a}^\dagger, \tag{5.18}$$

$$\hat{a}'^\dagger = \hat{S}(\xi)\hat{a}^\dagger\hat{S}^\dagger(\xi) = \nu^*\hat{a} + \mu\hat{a}^\dagger, \tag{5.19}$$

where $\mu = \cosh|\xi|$ and $\nu = e^{-\varphi_\xi}\sinh|\xi|$. This result follows from the Baker–Haussdorff lemma (5.7) with $\hat{B} \equiv \hat{a}$ (and $\hat{B} \equiv \hat{a}^\dagger$) and $z\hat{A} \equiv \frac{1}{2}[\xi^*\hat{a}^2 - \xi(\hat{a}^\dagger)^2]$. The minimal and maximal values of the quadrature fluctuations are $e^{-2|\xi|}$ and $e^{2|\xi|}$, as we will see later.

There are several different ways to produce squeezed states. We will see in a later lecture that fluorescent light from resonantly driven atoms is squeezed, and that effective nonlinear atom-light interactions lead to squeezing.

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Supplement to Lecture 5: Baker–Campbell–Haussdorff formula

We want to prove the relation (5.2),

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]},$$

which is valid for two operators $\hat{A}$ and $\hat{B}$ whose double commutators vanish, i.e. $[[\hat{A},\hat{B}],\hat{A}] = [[\hat{A},\hat{B}],\hat{B}] = 0$. First, we define an exponential operator $\hat{E}(z) = e^{z(\hat{A}+\hat{B})}$. We want to write this operator as $\hat{E}(z) = \hat{E}_1(z)\hat{E}_2(z)$ where $\hat{E}_1(z) = e^{z\hat{A}}$. Hence, $\hat{E}_2(z) = \hat{E}_1^{-1}(z)\hat{E}(z) = e^{-z\hat{A}}e^{z(\hat{A}+\hat{B})}$. From these definitions, we find that $\hat{E}_2(z)$ fulfils the following differential equation:

$$\frac{d}{dz}\hat{E}_2(z) = e^{-z\hat{A}}\hat{B}e^{z(\hat{A}+\hat{B})} = \left[\hat{E}_1^{-1}(z)\hat{B}\hat{E}_1(z)\right]\hat{E}_2(z) := \hat{B}'(z)\hat{E}_2(z)$$

with the initial condition $\hat{E}_2(0) = 1$. Before solving this equation, let us first look at the differential equation satisfied by $\hat{B}'(z)$ which reads

$$\frac{d}{dz}\hat{B}'(z) = \hat{E}_1^{-1}(z)\left[\hat{B},\hat{A}\right]\hat{E}_1(z) = -c,$$

where we have used the definition $[\hat{A},\hat{B}] = c$ and the assumption that this commutator is a c-number. The solution to this differential equation with the initial condition $\hat{B}'(0) = \hat{B}$ is simply $\hat{B}'(z) = \hat{B} - cz$. Hence, the solution for $\hat{E}_2(z)$ becomes

$$\hat{E}_2(z) = e^{z\hat{B}-\frac{c}{2}z^2} = e^{z\hat{B}}e^{-\frac{z^2}{2}[\hat{A},\hat{B}]},$$

which, when inserted into the definition of $\hat{E}(z)$ for $z = 1$, becomes Eq. (5.2).

The Baker–Haussdorff lemma (5.7) is best checked by Taylor expansion with respect to z:

$$e^{z\hat{A}}\hat{B}e^{-z\hat{A}} = \left(1 + z\hat{A} + \frac{z^2}{2!}\hat{A}^2 + \dots\right)\hat{B}\left(1 - z\hat{A} + \frac{z^2}{2!}\hat{A}^2 + \dots\right)$$

$$= \hat{B} + z\left(\hat{A}\hat{B} - \hat{B}\hat{A}\right) + \frac{z^2}{2!}\left(\hat{A}^2\hat{B} + \hat{B}\hat{A}^2 - 2\hat{A}\hat{B}\hat{A}\right) + \dots$$

$$= \hat{B} + z\left[\hat{A},\hat{B}\right] + \frac{z^2}{2!}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right] + \dots$$