

Lecture 6: Quantum states in phase space

- classical vs. quantum statistics, quasi-probability distributions
- operator expansion in phase space

Classical vs. quantum statistics, quasi-probability distributions: The phase-space picture we have developed in the last lecture for coherent and squeezed states is not quite correct as it cannot account for number states (in fact, all states that are not of Gaussian form). Here we will derive a proper phase-space picture that allows to associate to each quantum state, pure or mixed, a unique c -number function in phase space that contains all information about the quantum state under consideration.

Classical average: For this purpose, let us first recall that the classical statistical average of a classical quantity X is obtained by integrating the values of X in the phase space spanned by the complex number α , $X(\alpha)$, with a classical probability distribution function

$$\langle X \rangle_{\text{cl}} = \int d^2\alpha P_{\text{cl}}(\alpha) X(\alpha). \quad (6.1)$$

The classical probability distribution function $P_{\text{cl}}(\alpha)$ fulfills all requirements of a probability density, that is, it is non-negative for all values of α , $P_{\text{cl}}(\alpha) \geq 0$, and it is normalized to unity (Fig. 5). Our goal is to find a relation that bears some similarity to (6.1) for quantum states.

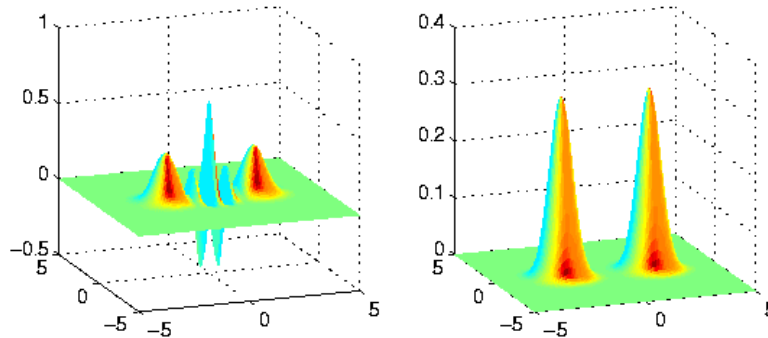


FIG. 5: Two examples of phase-space distributions. The distribution on the left could be interpreted as a classical probability distribution, the distribution on the right cannot. However, it is a perfectly valid quantum-mechanical quasi-probability distribution.

Quantum average: Recall that the quantum-mechanical expectation value of an operator \hat{O} in a pure quantum state $|\psi\rangle$ is calculated as

$$\langle \hat{O} \rangle_\psi = \langle \psi | \hat{O} | \psi \rangle.$$

The quantum state itself, however, could be a (classical) mixture of different pure states $|\psi_i\rangle$, each being associated with a probability $p_i \geq 0$. For example, we might not be exactly sure which state our preparation procedure had actually realized. We might have had one photon in a given mode (with probability p_1), or perhaps none at all (with probability $p_0 = 1 - p_1$). We would then calculate the expectation value of \hat{O} as $\langle \hat{O} \rangle = p_0 \langle 0 | \hat{O} | 0 \rangle + p_1 \langle 1 | \hat{O} | 1 \rangle$. In general, we assign to a mixed quantum state a statistical operator (density operator) $\hat{\rho} = \sum_i p_i |\psi_i\rangle \langle \psi_i|$. With the help of the density operator, the expectation value is now

$$\langle \hat{O} \rangle_\rho = \text{Tr} [\hat{\rho} \hat{O}], \quad (6.2)$$

where Tr denotes the operator trace.

Let us now assume that the quantum-mechanical operator \hat{O} is a functional of the photonic amplitude operators, $\hat{O} \equiv \hat{O}(\hat{a}, \hat{a}^\dagger)$. The classical analogue of this operator would be formally obtained if we replaced the operators \hat{a} and \hat{a}^\dagger by the complex amplitudes α and α^* , thereby obtaining a function $O(\alpha, \alpha^*)$. We have to be careful how we perform this replacement since we can write the same operator $\hat{O}(\hat{a}, \hat{a}^\dagger)$ in different operator orderings as \hat{a} and \hat{a}^\dagger do not commute. For example, consider the operator $\hat{O}(\hat{a}, \hat{a}^\dagger) = [\hat{x}(\varphi)]^2$ where $\hat{x}(\varphi) = \hat{a}e^{i\varphi} + \hat{a}^\dagger e^{-i\varphi}$ is the quadrature operator. Using the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, we can write it in these three different forms:

$$\begin{aligned} [\hat{x}(\varphi)]^2 &= \hat{a}^2 e^{2i\varphi} + (\hat{a}^\dagger)^2 e^{-2i\varphi} + (\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) && \text{symmetric order,} \\ &= \hat{a}^2 e^{2i\varphi} + (\hat{a}^\dagger)^2 e^{-2i\varphi} + (2\hat{a}^\dagger\hat{a} + 1) && \text{normal order,} \\ &= \hat{a}^2 e^{2i\varphi} + (\hat{a}^\dagger)^2 e^{-2i\varphi} + (2\hat{a}\hat{a}^\dagger - 1) && \text{anti-normal order.} \end{aligned} \quad (6.3)$$

All three expressions represent exactly the same operator, but their form differs. In normal order, all creation operators are to the left of the annihilation operators, in anti-normal order the creation operators are to the right of them, and in symmetric order they appear in symmetrized form. From now on we will concentrate on symmetric operator order.

We now return to the classical function $O(\alpha, \alpha^*)$ and rewrite it identically as

$$O(\alpha, \alpha^*) = \int d^2\beta \delta(\alpha - \beta) O(\beta, \beta^*), \quad (6.4)$$

where $\delta(\alpha)$ is the standard two-dimensional δ function of the complex variable $\alpha = \alpha' + i\alpha''$ which we can write in these equivalent forms:

$$\begin{aligned}\delta(\alpha) &= \delta(\alpha')\delta(\alpha'') = \frac{1}{(2\pi)^2} \int dx dy e^{i(\alpha'x + \alpha''y)} \\ &= \frac{1}{\pi^2} \int d^2\gamma e^{\alpha\gamma^* - \alpha^*\gamma} = \frac{1}{\pi^2} \int d^2\gamma e^{\alpha^*\gamma - \alpha\gamma^*},\end{aligned}\quad (6.5)$$

where we have defined $\gamma = -(y - ix)/2$. Inserting the second form of Eq. (6.5) into Eq. (6.4), we obtain

$$O(\alpha, \alpha^*) = \frac{1}{\pi^2} \int d^2\beta d^2\gamma e^{(\alpha^* - \beta^*)\gamma - (\alpha - \beta)\gamma^*} O(\beta, \beta^*). \quad (6.6)$$

If we now formally replace the complex amplitudes α and α^* by the operators \hat{a} and \hat{a}^\dagger , we obtain the operator expansion of $\hat{O}(\hat{a}, \hat{a}^\dagger)$ in symmetric order as

$$\hat{O}(\hat{a}, \hat{a}^\dagger) = \frac{1}{\pi^2} \int d^2\beta d^2\gamma e^{(\hat{a}^\dagger - \beta^*)\gamma - (\hat{a} - \beta)\gamma^*} O(\beta, \beta^*). \quad (6.7)$$

The operator

$$\hat{\delta}(\hat{a} - \beta) = \frac{1}{\pi^2} \int d^2\gamma e^{(\hat{a}^\dagger - \beta^*)\gamma - (\hat{a} - \beta)\gamma^*} = \frac{1}{\pi^2} \int d^2\gamma e^{\gamma^*\beta - \gamma\beta^*} \hat{D}(\gamma) \quad (6.8)$$

is just the Fourier transform of the displacement operator $\hat{D}(\gamma)$, and it is called the operator-valued δ function in symmetric order. Finally, if we take the expectation value of both sides of Eq. (6.7), we find that

$$\langle \hat{O}(\hat{a}, \hat{a}^\dagger) \rangle = \int d^2\beta W(\beta) O(\beta, \beta^*), \quad (6.9)$$

where

$$\boxed{W(\beta) = \langle \hat{\delta}(\hat{a} - \beta) \rangle = \text{Tr} \left[\hat{\rho} \hat{\delta}(\hat{a} - \beta) \right]} \quad (6.10)$$

is called the *Wigner function*.

Equation (6.9) has to be compared with the classical equivalent (6.1). Both expressions look formally the same. The expectation value of both classical and quantum observables are obtained by averaging the values of the observable in phase space with a distribution function. The important difference is that in the quantum case the Wigner function is in general not a well-defined probability density as it might become negative. However, the Wigner function is still properly normalized to unity. Yet, it can be used to obtain expectation values of operators, and is thus called a quasi-probability distribution.

Operator expansion in phase space: Until now, the analogy between Eqs. (6.1) and (6.9) was rather formal as we have no precise means of computing the classical function $O(\alpha, \alpha^*)$. In order to do so, we effectively need to invert Eq. (6.7). For this purpose, we evaluate $\text{Tr}[\hat{\delta}(\hat{a} - \beta)\hat{\delta}(\hat{a} - \gamma)]$ in the coherent-state basis which we write as

$$\text{Tr}[\hat{\delta}(\hat{a} - \beta)\hat{\delta}(\hat{a} - \gamma)] = \frac{1}{\pi^4} \int d^2\alpha_1 d^2\alpha_2 e^{\alpha_1\beta^* - \alpha_1^*\beta} e^{\alpha_2\gamma^* - \alpha_2^*\gamma} \text{Tr}[\hat{D}(\alpha_1)\hat{D}(\alpha_2)]. \quad (6.11)$$

First we note that we only have to compute $\text{Tr}[\hat{D}(\alpha_1)\hat{D}(\alpha_2)]$ which yields

$$\begin{aligned} \text{Tr}[\hat{D}(\alpha_1)\hat{D}(\alpha_2)] &= \text{Tr} \left[e^{\alpha_1\hat{a}^\dagger} e^{-\alpha_1^*\hat{a}} e^{-\alpha_2^*\hat{a}} e^{\alpha_2\hat{a}^\dagger} \right] e^{\frac{1}{2}(|\alpha_2|^2 - |\alpha_1|^2)} \\ &= \text{Tr} \left[e^{(\alpha_1 + \alpha_2)\hat{a}^\dagger} e^{-(\alpha_1^* + \alpha_2^*)\hat{a}} \right] e^{\frac{1}{2}(|\alpha_2|^2 - |\alpha_1|^2)} \\ &= \frac{1}{\pi} \int d^2\alpha \langle \alpha | e^{(\alpha_1 + \alpha_2)\hat{a}^\dagger} e^{-(\alpha_1^* + \alpha_2^*)\hat{a}} | \alpha \rangle e^{\frac{1}{2}(|\alpha_2|^2 - |\alpha_1|^2)} \\ &= \frac{1}{\pi} \int d^2\alpha e^{(\alpha_1 + \alpha_2)\alpha^* - (\alpha_1^* + \alpha_2^*)\alpha} e^{\frac{1}{2}(|\alpha_2|^2 - |\alpha_1|^2)} \\ &= \pi \delta(\alpha_1 + \alpha_2). \end{aligned} \quad (6.12)$$

Hence, $\text{Tr}[\hat{\delta}(\hat{a} - \beta)\hat{\delta}(\hat{a} - \gamma)] = \frac{1}{\pi} \delta(\beta - \gamma)$. This relation can be used to invert Eq. (6.7) as

$$O(\beta, \beta^*) = \pi \text{Tr} \left[\hat{O}(\hat{a}, \hat{a}^\dagger) \hat{\delta}(\hat{a} - \beta) \right] \quad (6.13)$$

and the operator expansion can be written as

$$\hat{O}(\hat{a}, \hat{a}^\dagger) = \pi \int d^2\alpha \text{Tr} \left[\hat{O}(\hat{a}, \hat{a}^\dagger) \hat{\delta}(\hat{a} - \alpha) \right] \hat{\delta}(\hat{a} - \alpha). \quad (6.14)$$

Since Eq. (6.14) holds for all operators, it can also be used to expand the density operator itself. Recalling that $\text{Tr}[\hat{\rho}\hat{\delta}(\hat{a} - \alpha)] = W(\alpha)$, we obtain

$$\boxed{\hat{\rho} = \pi \int d^2\alpha W(\alpha) \hat{\delta}(\hat{a} - \alpha)} \quad (6.15)$$

which is an expansion of the density operator in terms of the operator-valued δ function in symmetric order and the Wigner function.

Equations (6.10) and (6.15) are effectively mutual inversions. We can interpret this result as a one-to-one correspondence between the density operator $\hat{\rho}$ and the (c -number) phase-space function $W(\alpha)$. Both contain exactly the same information about the quantum state of the system. The knowledge of the Wigner function of a quantum state is therefore enough to specify the quantum state completely.

Lecture 7: Examples of Wigner functions

- Number states
- coherent states and squeezed states
- calculation of expectation values using Wigner functions

Wigner function of a number state: In the previous lecture we have seen that the Wigner function contains all the information of a given quantum state. In other words, it is an effective tool for visualization quantum states. Let us first consider the number state $|n\rangle$. The density operator associated with it is clearly $\hat{\rho} = |n\rangle\langle n|$. Recall from Eq. (6.10) that the Wigner function is defined as $W(\alpha) = \text{Tr}[\hat{\rho}\delta(\hat{a} - \alpha)]$. Hence, we have to compute

$$W_n(\alpha) = \frac{1}{\pi^2} \int d^2\beta e^{\alpha\beta^* - \alpha^*\beta} \langle n|\hat{D}(\beta)|n\rangle, \quad (7.1)$$

where we have used the definition of the operator-valued δ function [Eq. (6.8)]. The expectation value of the displacement operator in the number state can be evaluated as follows:

$$\begin{aligned} \langle n|\hat{D}(\beta)|n\rangle &= e^{-\frac{1}{2}|\beta|^2} \langle n|e^{\beta\hat{a}^\dagger}e^{-\beta^*\hat{a}}|n\rangle \\ &= e^{-\frac{1}{2}|\beta|^2} \sum_{m=0}^n \frac{(-|\beta|^2)^m}{(m!)^2} \langle n|(\hat{a}^\dagger)^m\hat{a}^m|n\rangle \\ &= e^{-\frac{1}{2}|\beta|^2} \sum_{m=0}^n \frac{(-|\beta|^2)^m}{m!} \binom{n}{m} = e^{-\frac{1}{2}|\beta|^2} L_n(|\beta|^2). \end{aligned} \quad (7.2)$$

The function $L_n(x)$ is the Laguerre polynomial of order n . The Fourier transform of it turns out to be also a Laguerre polynomial of the same order with the result that

$$\boxed{W_n(\alpha) = \frac{2}{\pi} (-1)^n e^{-2|\alpha|^2} L_n(4|\alpha|^2)}. \quad (7.3)$$

What we see from Eq. (7.3) is that for odd photon numbers the Wigner function becomes negative around the origin of phase space because $L_n(0) = 1$. Moreover, for all $n > 0$ the Wigner function obtains negative values somewhere in phase space. This means that we cannot interpret the Wigner function as a classical probability distribution which should be non-negative everywhere. We also see that the Wigner function is bounded by

$$-\frac{2}{\pi} \leq W(\alpha) \leq \frac{2}{\pi}. \quad (7.4)$$

This is generally true for Wigner functions of any quantum state which we will not prove.

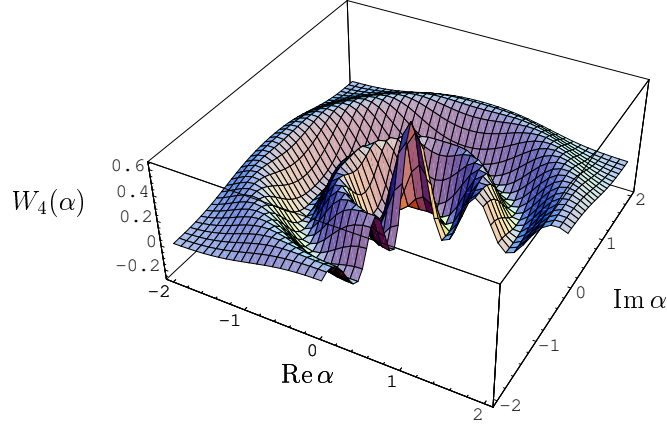


FIG. 6: Wigner function of a number state with 4 photons.

An example of the Wigner function for $n = 4$ is shown in the Fig. 6. We see that it is rotationally symmetric as it is clear from Eq. (7.3) that the complex amplitude appears only in the form $|\alpha|^2$.

Wigner function of coherent states and squeezed states: As an example of a non-negative quasi-probability distribution, we compute the Wigner function of a coherent state $|\alpha_0\rangle$. Since this is also a pure state, the density operator is simply $\hat{\rho} = |\alpha_0\rangle\langle\alpha_0|$. The calculation is rather straightforward and we obtain

$$\begin{aligned} W(\alpha) &= \frac{1}{\pi^2} \int d^2\beta e^{\alpha\beta^* - \alpha^*\beta} \langle\alpha_0|\hat{D}(\beta)|\alpha_0\rangle \\ &= \frac{1}{\pi^2} \int d^2\beta e^{(\alpha-\alpha_0)\beta^* - (\alpha^*-\alpha_0^*)\beta} e^{-\frac{1}{2}|\beta|^2} = \frac{2}{\pi} e^{-2|\alpha-\alpha_0|^2}. \end{aligned} \quad (7.5)$$

Again, the Wigner function is bounded by $2/\pi$, but this time it is a strictly positive function, namely a Gaussian function centred at α_0 (see Fig. 7). If we recall the naive phase-space picture derived after Eq. (5.16) we see that we can associate the uncertainty area with the area determined by the width at half maximum of the Gaussian function (7.5).

In order to see that even better, let us look at squeezed states. In particular, we will compute the Wigner function of the squeezed vacuum state $|\xi\rangle = \hat{S}(\xi)|0\rangle$. To do so, we will need the expectation value $\langle\xi|\hat{D}(\beta)|\xi\rangle = \langle 0|\hat{S}^\dagger(\xi)\hat{D}(\beta)\hat{S}(\xi)|0\rangle$. Recall that the squeeze operator transforms the photonic amplitude operators as $\hat{a}' = \hat{S}(\xi)\hat{a}\hat{S}^\dagger(\xi) = \mu\hat{a} + \nu\hat{a}^\dagger$ and $\hat{a}'^\dagger = \hat{S}(\xi)\hat{a}^\dagger\hat{S}^\dagger(\xi) = \nu^*\hat{a} + \mu^*\hat{a}^\dagger$ where $\mu = \cosh|\xi|$ and $\nu = e^{i\varphi_\nu} \sinh|\xi|$ [see Eqs. (5.18) and (5.19)]. If we expand $\hat{D}(\beta)$ in a Taylor series with respect to \hat{a} and \hat{a}^\dagger and insert identity operators $\hat{S}(\xi)\hat{S}^\dagger(\xi) = \hat{I}$ between all operator products, it is easy to see that $\hat{S}^\dagger(\xi)\hat{D}(\beta)\hat{S}(\xi)$

$= \hat{D}(\beta\mu + \beta^*\nu)$ and hence

$$W(\alpha) = \frac{1}{\pi^2} \int d^2\beta e^{\alpha\beta^* - \alpha^*\beta} e^{-\frac{1}{2}|\beta\mu + \beta^*\nu|^2}. \quad (7.6)$$

Assuming $\xi \in \mathbb{R}$ and performing the final integration yields ($\alpha = \alpha' + i\alpha''$)

$$W(\alpha) = \frac{2}{\pi} \exp \left[-2 \left(\frac{\alpha'^2}{e^{-2\xi}} + \frac{\alpha''^2}{e^{2\xi}} \right) \right]. \quad (7.7)$$

The Wigner function (7.7) is again a Gaussian function, but this time centred at the origin of phase space (see Fig. 8). In contrast to Eq. (7.5) it is not rotationally symmetric but

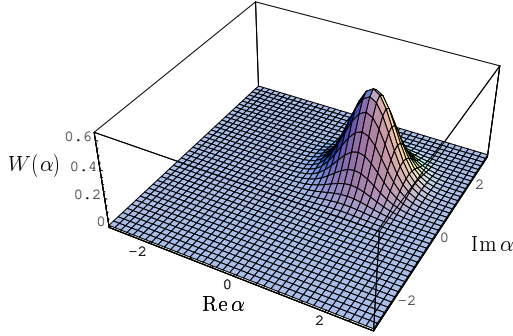


FIG. 7: Wigner function of a coherent state with $\alpha_0 = 1 + i$.

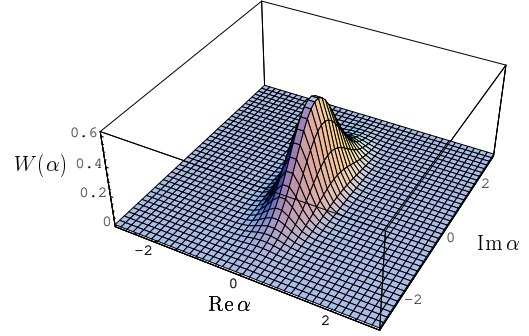


FIG. 8: Wigner function of a squeezed vacuum state with $\xi = 1$.

rather a product of two Gaussian functions with widths $e^{-2\xi}$ (along the real axis in phase space) and $e^{2\xi}$ (along the imaginary axis). This justifies the naive phase space picture for Gaussian states. However, number states and other non-Gaussian states cannot sensibly be described by the naive picture, and a thorough analysis of the Wigner function is needed.

Calculation of expectation values using Wigner functions:

We can use the Wigner function as in classical probability theory to compute expectation values of (symmetrically ordered) operators as

$$\langle \hat{O}(\hat{a}, \hat{a}^\dagger) \rangle_{\text{sym}} = \int d^2\alpha O(\alpha, \alpha^*) W(\alpha). \quad (7.8)$$

As an example, we take the operator $\hat{O}(\hat{a}, \hat{a}^\dagger)$ to be the variance of the quadrature operator $\hat{x}(\varphi)$, $\hat{O}(\hat{a}, \hat{a}^\dagger) = [\Delta\hat{x}(\varphi)]^2$ which, as we have previously see, is proportional to the variance of the electric-field strength. Since $\langle [\Delta\hat{x}(\varphi)]^2 \rangle = \langle \hat{x}^2(\varphi) \rangle - \langle \hat{x}(\varphi) \rangle^2$, we have to compute

$$\langle \hat{O}(\hat{a}, \hat{a}^\dagger) \rangle_{\text{sym}} = [\langle \hat{a}^2 \rangle e^{2i\varphi} + \langle (\hat{a}^\dagger)^2 \rangle e^{-2i\varphi} + \langle \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} \rangle] - [\langle \hat{a} \rangle e^{i\varphi} + \langle \hat{a}^\dagger \rangle e^{-i\varphi}]^2. \quad (7.9)$$

First, let us look at the coherent state $|\alpha_0\rangle$ whose Wigner function is given by Eq. (7.5). The necessary integrations are therefore Gaussian integrals of the form

$$I_{mn} = \frac{2}{\pi} \int d^2\alpha \alpha^m \alpha^{*n} e^{-2|\alpha - \alpha_0|^2}, \quad (7.10)$$

with which we can rewrite Eq. (7.9) as

$$\langle \hat{O}(\hat{a}, \hat{a}^\dagger) \rangle_{\text{sym}} = [I_{20}e^{2i\varphi} + I_{02}e^{-2i\varphi} + 2I_{11}] - [I_{10}e^{i\varphi} + I_{01}e^{-i\varphi}]^2. \quad (7.11)$$

Using $I_{10} = \alpha_0$, $I_{01} = \alpha_0^*$, $2I_{11} = 1 + 2|\alpha_0|^2$, $I_{20} = \alpha_0^2$, and $I_{02} = \alpha_0^{*2}$, we find that

$$\langle \alpha_0 | [\Delta \hat{x}(\varphi)]^2 | \alpha_0 \rangle = 1, \quad (7.12)$$

which is, apart from the factor $\omega^2 |A_k(\mathbf{r})|^2$, just Eq. (5.13).

As a second example, we consider the squeezed vacuum state $|\xi\rangle$ whose Wigner function (7.7) is also a Gaussian. The relevant integrals are now

$$J_{mn} = \frac{2}{\pi} \int d^2\alpha \alpha^m \alpha^{*n} \exp \left[-2 \left(\frac{\alpha'^2}{e^{-2\xi}} + \frac{\alpha''^2}{e^{2\xi}} \right) \right]. \quad (7.13)$$

Without calculation, we know that $J_{10} = J_{01} = 0$ because the Wigner function is centred at the origin in phase space. The other relevant integrals are $2J_{11} = \cosh 2\xi$ and $J_{20} = J_{02} = -\frac{1}{2} \sinh 2\xi$. Combining these integrals, we obtain

$$\langle \xi | [\Delta \hat{x}(\varphi)]^2 | \xi \rangle = \cosh 2\xi - \frac{1}{2} \sinh 2\xi (e^{2i\varphi} + e^{-2i\varphi}), \quad (7.14)$$

which is seen to depend both on the squeezing parameter ξ and on the phase φ . The minimum of Eq. (7.14) is obtained for $\varphi = 0$, and the maximum for $\varphi = \pi/2$ with the values

$$\langle \xi | [\Delta \hat{x}(0)]^2 | \xi \rangle = e^{-2\xi}, \quad (7.15)$$

$$\langle \xi | [\Delta \hat{x}(\frac{\pi}{2})]^2 | \xi \rangle = e^{2\xi}, \quad (7.16)$$

which is exactly what we claimed in an earlier lecture. In most cases, and in particular for Gaussian states, it turns out to be simpler to study expectation values using the Wigner function than by direct calculation.